

# On the toric ideal of a matroid

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**ABSTRACT.** Describing minimal generating sets of toric ideals is a well-studied and difficult problem. In 1980 White conjectured that the toric ideal associated to a matroid is generated by quadratic binomials corresponding to single element symmetric basis exchanges. We prove White's conjecture for strongly base orderable matroids.

Furthermore, we prove for arbitrary matroids a weak version of White's conjecture. It asserts that both the ideal associated to a matroid and the one generated by quadratic binomials corresponding to single element symmetric basis exchanges define the same projective scheme. In other words their saturations with respect to the ideal generated by all variables are equal.

## 1. Introduction

Let  $M$  be a matroid on a ground set  $E$  with the set of bases  $\mathfrak{B} \subset \mathcal{P}(E)$  (the reader is referred to [18] for background of matroid theory). For a fixed field  $\mathbb{K}$  let  $S_M := \mathbb{K}[y_B : B \in \mathfrak{B}]$  be a polynomial ring. There is a natural  $\mathbb{K}$ -homomorphism:

$$\varphi_M : S_M \ni y_B \rightarrow \prod_{e \in B} x_e \in \mathbb{K}[x_e : e \in E].$$

The toric ideal of a matroid  $M$ , denoted by  $I_M$ , is the kernel of  $\varphi_M$ .

The family of bases  $\mathfrak{B}$ , just from the definition of a matroid, is nonempty and satisfies *exchange property*:

*For every bases  $B_1, B_2$  and  $e \in B_1 \setminus B_2$  there is  $f \in B_2 \setminus B_1$ , such that  $B_1 \cup f \setminus e$  is also a basis.*

As it was shown by Brualdi [2], the following *symmetric exchange property* also holds:

*For every bases  $B_1, B_2$  and  $e \in B_1 \setminus B_2$  there is  $f \in B_2 \setminus B_1$ , such that both  $B_1 \cup f \setminus e$  and  $B_2 \cup e \setminus f$  are also bases.*

Surprisingly a stronger property, known as *multiple symmetric exchange property*, is true (for simple proofs see [16, 26], and [14, 15] for more exchange properties):

*For every bases  $B_1, B_2$  and  $A_1 \subset B_1$  there is  $A_2 \subset B_2$ , such that both  $B_1 \cup A_2 \setminus A_1$  and  $B_2 \cup A_1 \setminus A_2$  are also bases.*

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We say that a pair of bases  $D_1, D_2$  is obtained from a pair of bases  $B_1, B_2$  by a *double swap* if  $D_1 = B_1 \cup f \setminus e$  and  $D_2 = B_2 \cup e \setminus f$  for some  $e \in B_1$  and  $f \in B_2$ . Then clearly the corresponding quadratic binomial

$$y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$$

belongs to the ideal  $I_M$ .

In 1980 White conjectured [23] that for every matroid  $M$  its toric ideal  $I_M$  is generated by quadratic binomials corresponding to double swaps.

**CONJECTURE 1.** (White) *For every matroid  $M$  its toric ideal  $I_M$  is generated by quadratic binomials  $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$  for bases  $D_1, D_2$  obtained from bases  $B_1, B_2$  by a double swap.*

The most significant partial result is due to Blasiak [1], who confirmed the conjecture for graphical matroids. Kashiwabara [13] solved the case of matroids of rank at most 3. Schweig [20] proved the case of lattice path matroids, which are a subclass of transversal matroids.

A matroid is *strongly base orderable* if for any two bases  $B_1$  and  $B_2$  there is a bijection  $\pi : B_1 \rightarrow B_2$  with the multiple symmetric exchange property, that is  $B_1 \cup \pi(A) \setminus A$  is a basis for any  $A \subset B_1$ . This implies that also  $B_2 \cup A \setminus \pi(A)$  is a basis for any  $A \subset B_1$ , moreover we can assume that  $\pi$  is the identity on  $B_1 \cap B_2$ . The class of strongly base orderable matroids is closed under taking minors. It is already a large class of matroids, characterized by a certain property instead of a specific presentation (like it is for graphical, transversal or lattice path matroids).

We prove White's conjecture for strongly base orderable matroids. As a consequence it is true for gammoids (every gammoid is strongly base orderable [19]), and in particular for transversal matroids (every transversal matroid is a gammoid [18]). So far, for transversal matroids, it was known only that the toric ideal  $I_M$  is generated by quadratic binomials [4].

**THEOREM 2.** *If  $M$  is a strongly base orderable matroid, then the toric ideal  $I_M$  is generated by quadratic binomials  $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$  for bases  $D_1, D_2$  obtained from bases  $B_1, B_2$  by a double swap.*

Our argument uses an idea of Davies and McDiarmid [7] from the proof of theorem asserting that if the ground set  $E$  of two strongly base orderable matroids can be partitioned into  $n$  bases, then there exists also a common partition.

Describing minimal generating set of a toric ideal is a natural problem in algebraic combinatorics. From the point of view of algebraic geometry it is natural to study a projective (as the ideal  $I_M$  is homogeneous because every basis has the same number of elements) toric variety  $Y_M = \text{Proj}(S_M/I_M)$  (we refer the reader to [8, 5] for background of toric geometry). It is often required that projective toric variety is normal. White proved that indeed the variety  $Y_M$  is even projectively normal [24].

Let  $J_M$  be the ideal generated by quadratic binomials  $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$  for bases  $D_1, D_2$  obtained from bases  $B_1, B_2$  by a double swap. Clearly,  $J_M \subset I_M$ . White's conjecture asserts that  $J_M = I_M$ , which in other words means that ideals  $I_M$  and  $J_M$  define the same affine scheme. We prove for arbitrary matroids a weak version of White's conjecture. It asserts that ideals  $I_M$  and  $J_M$  define the same projective scheme (cf. [17, Section 4]).

**THEOREM 3.** *For arbitrary matroids projective schemes  $\text{Proj}(S_M/I_M)$  and  $\text{Proj}(S_M/J_M)$  are equal.*

Let  $\mathfrak{m}$  be the ideal generated by all variables in  $S_M$  (so called irrelevant ideal). Recall that  $I : \mathfrak{m}^\infty = \{a \in S_M : a\mathfrak{m}^n \subset I \text{ for some } n \in \mathbb{N}\}$  is called the *saturation* of an ideal  $I$  with respect to the ideal  $\mathfrak{m}$ . Algebraic reformulation of the fact that ideals  $I_M$  and  $J_M$  define the same projective scheme, is that their saturations with respect to  $\mathfrak{m}$  are equal, that is  $I_M : \mathfrak{m}^\infty = J_M : \mathfrak{m}^\infty$ . In particular their radicals are equal.

Conjecture 1 is an algebraic reformulation of the original one, which was expressed in combinatorial language (cf. [22]). In fact White defined three properties of a matroid of growing difficulty, and conjectured that all matroids satisfy them. The first property asserts that the toric ideal  $I_M$  is generated by quadratic binomials, the second one is Conjecture 1 for  $M$ , while the most difficult is an analog of Conjecture 1 for noncommutative polynomial ring. We discuss them in details in the last section. We prove that Conjecture 1 holds for  $M \oplus M$  if and only if its noncommutative version holds for  $M$ . In particular we get that the strongest property holds for all strongly base orderable, graphical, and cographical matroids. We mention also how to extend Theorems 2 and 3 to discrete polymatroids.

## 2. White's conjecture for strongly base orderable matroids

**PROOF OF THEOREM 2.** Because the ideal  $I_M$  is toric, it is generated by binomials. Thus it is enough to prove that quadratic binomials corresponding to double swaps generate all binomials of  $I_M$ .

Fix  $n \geq 2$ . We are going to show by decreasing induction on an overlap function

$$d(y_{B_1} \cdots y_{B_n}, y_{D_1} \cdots y_{D_n}) = \min_{\pi \in S_n} \sum_i |B_i \cap D_{\pi(i)}|$$

that a binomial  $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$  is generated by quadratic binomials corresponding to double swaps. Clearly, the biggest value of  $d$  is  $r(M)n$ , where  $r(M)$  is the cardinality of a basis of  $M$ .

If  $d(y_{B_1} \cdots y_{B_n}, y_{D_1} \cdots y_{D_n}) = r(M)n$ , then there exists a permutation  $\pi \in S_n$  such that for each  $i$  holds  $B_i = D_{\pi(i)}$ , so  $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} = 0$ .

Suppose the assertion is true for all binomials with overlap function greater than  $d < r(M)n$ . Let  $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n}$  be a binomial of  $I_M$  with the overlap function of monomials equal to  $d$ . Then there exists  $i$  such that there is  $e \in B_i \setminus D_i$ . Clearly,  $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$  if and only if  $B_1 \cup \cdots \cup B_n = D_1 \cup \cdots \cup D_n$  as multisets, thus there exists  $j \neq i$  such that  $e \in D_j \setminus B_j$ . Without loss of generality we can assume that  $i = 1, j = 2$ . Since  $M$  is a strongly base orderable matroid, there exist bijections  $\pi_B : B_1 \rightarrow B_2$  and  $\pi_D : D_1 \rightarrow D_2$  with the multiple symmetric exchange property. We can assume that  $\pi_B$  is the identity on  $B_1 \cap B_2$ , and similarly that  $\pi_D$  is the identity on  $D_1 \cap D_2$ .

Let  $G$  be a graph on a vertex set  $B_1 \cup B_2 \cup D_1 \cup D_2$  with a set of edges  $\{b, \pi_B(b)\}$  for  $b \in B_1 \setminus B_2$  and  $\{d, \pi_D(d)\}$  for  $d \in D_1 \setminus D_2$ . Graph  $G$  is bipartite since it is a sum of two matchings. Split the vertex set into two independent sets  $S$  and  $T$ . Define:

$$B'_1 = (S \cap (B_1 \cup B_2)) \cup (B_1 \cap B_2), \quad B'_2 = (T \cap (B_1 \cup B_2)) \cup (B_1 \cap B_2),$$

$$D'_1 = (S \cap (D_1 \cup D_2)) \cup (D_1 \cap D_2), \quad D'_2 = (T \cap (D_1 \cup D_2)) \cup (D_1 \cap D_2).$$

By the multiple symmetric exchange property of  $\pi_B$  sets  $B'_1, B'_2$  are bases, and they are obtained from the pair  $B_1, B_2$  by a sequence of double swaps. Therefore the binomial  $y_{B_1}y_{B_2}y_{B_3} \cdots y_{B_n} - y_{B'_1}y_{B'_2}y_{B_3} \cdots y_{B_n}$  is generated by quadratic binomials corresponding to double swaps. Analogously the binomial  $y_{D_1}y_{D_2}y_{D_3} \cdots y_{D_n} - y_{D'_1}y_{D'_2}y_{D_3} \cdots y_{D_n}$  is generated by quadratic binomials corresponding to double swaps. Moreover, since  $S$  and  $T$  are disjoint we have that

$$d(y_{B'_1}y_{B'_2}y_{B_3} \cdots y_{B_n}, y_{D'_1}y_{D'_2}y_{D_3} \cdots y_{D_n}) > d(y_{B_1}y_{B_2}y_{B_3} \cdots y_{B_n}, y_{D_1}y_{D_2}y_{D_3} \cdots y_{D_n})$$

thus by inductive assumption  $y_{B'_1}y_{B'_2}y_{B_3} \cdots y_{B_n} - y_{D'_1}y_{D'_2}y_{D_3} \cdots y_{D_n}$  is generated by quadratic binomials corresponding to double swaps. By adding three binomials we get the inductive assertion.  $\square$

### 3. Projective scheme-theoretic version of White's conjecture for arbitrary matroids

Let us recall that  $I_M$  is the ideal associated to a matroid and  $J_M$  is the ideal generated by quadratic binomials  $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$  for bases  $D_1, D_2$  obtained from bases  $B_1, B_2$  by a double swap.

In order to get a weaker version of White's conjecture instead of comparing ideals  $I_M$  and  $J_M$  we compare schemes defined by them. A homogeneous ideal defines two schemes – affine and projective. It is well-known that two ideals in the ring of polynomials  $S_M = \mathbb{K}[x_B : B \in \mathfrak{B}]$  define the same affine scheme if and only if they are equal. Thus White's conjecture asserts equality of affine schemes of  $I_M$  and  $J_M$ . Homogeneous ideals  $I_M$  and  $J_M$  define the same projective scheme if and only if their saturations with respect to the maximal ideal  $\mathfrak{m} = (x_B : B \in \mathfrak{B})$  are equal. That is in order to prove Theorem 3 we will show that  $I_M : \mathfrak{m}^\infty = J_M : \mathfrak{m}^\infty$ . Since both  $I_M$  and  $J_M$  are contained in  $\mathfrak{m}$ , this implies that both ideals have equal radicals. In particular they have the same affine set of zeros.

PROOF OF THEOREM 3. Clearly, the ideal  $I_M$  is saturated, that is  $I_M : \mathfrak{m}^\infty = I_M$ . Since  $J_M \subset I_M$  we know that  $J_M : \mathfrak{m}^\infty \subset I_M : \mathfrak{m}^\infty = I_M$ .

To prove inclusion  $I_M \subset J_M : \mathfrak{m}^\infty$  it is enough to consider only binomials of  $I_M$  since  $I_M$ , as any toric ideal, is generated by binomials. To prove that a binomial  $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$  is in  $J_M : \mathfrak{m}^\infty$  it is enough to show that for each  $B \in \mathfrak{B}$  holds

$$y_B^{(r(M)-1)n} (y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n}) \in J_M,$$

where  $r(M)$  is the cardinality of a basis of  $M$ . It is because then

$$(y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n}) \mathfrak{m}^{(r(M)-1)n|\mathfrak{B}|} \in J_M.$$

Let  $B \in \mathfrak{B}$  be a basis. The polynomial ring  $S_M$  has a natural grading given by degree function  $\deg(y_{B'}) = 1$ , for each variable  $y_{B'}$ . We define also the  $B$ -degree by  $\deg_B(y_{B'}) = |B' \setminus B|$  (also for bases  $\deg_B(B') = |B' \setminus B|$ ). Notice that the ideal  $I_M$  is homogeneous with respect to both gradings. Additionally  $B$ -degree of  $y_B$  is zero, thus multiplying by  $y_B$  does not change the  $B$ -degree of a polynomial. Observe that if  $\deg_B(B') = 1$  then  $B'$  is made from  $B$  by a single symmetric exchange. We call such basis, and a corresponding variable, *balanced*. A monomial or a binomial is called *balanced* if it contains only balanced variables.

We prove by induction on  $n$  the following claim:

*If a binomial  $b \in I_M$  has  $B$ -degree at most  $n$  then  $y_B^{\deg_B(b) - \deg(b)} b \in J_M$ .*

As we argued before this will finish the proof, because  $B$ -degree of a single variable

is at most  $r(M) - 1$ . If  $\deg_B(b) - \deg(b) < 0$ , then the statement of the claim means that  $y_B^{\deg(b) - \deg_B(b)}$  divides  $b$ , and that the quotient belongs to  $J_M$ .

If  $n = 0$ , then the claim is obvious, since 0 is the only binomial of  $B$ -degree 0 in  $I_M$ .

Suppose  $n > 0$ , and  $b \in I_M$  is a binomial of  $B$ -degree  $n$ . It is easier to work with balanced variables, therefore we provide the following lemma.

LEMMA 4. *For every basis  $B' \in \mathfrak{B}$  there exist balanced bases  $B_1, \dots, B_{\deg_B(B')}$  such that*

$$y_B^{\deg_B(B')-1} y_{B'} - y_{B_1} \cdots y_{B_{\deg_B(B')}} \in J_M.$$

PROOF. The proof is by induction on  $\deg_B(B')$ . If  $\deg_B(B') > 1$  then for  $e \in B' \setminus B$  from the symmetric exchange property there exists  $f \in B$  such that both  $B_1 = B \cup e \setminus f$  and  $B'' = B' \cup f \setminus e$  are bases. Now  $\deg_B(B_1) = 1$  and  $\deg_B(B'') = \deg_B(B') - 1$ . Applying inductive assumption for  $B''$  we get balanced bases  $B_2, \dots, B_{\deg_B(B')}$  with

$$y_B^{\deg_B(B')-2} y_{B''} - y_{B_2} \cdots y_{B_{\deg_B(B')}} \in J_M.$$

Hence now

$$\begin{aligned} y_B^{\deg_B(B')-1} y_{B'} - y_{B_1} \cdots y_{B_{\deg_B(B')}} &= y_B^{\deg_B(B')-2} (y_B y_{B'} - y_{B_1} y_{B''}) + \\ &+ y_{B_1} (y_B^{\deg_B(B')-2} y_{B''} - y_{B_2} \cdots y_{B_{\deg_B(B')}}) \in J_M. \end{aligned}$$

□

Lemma 4 allows us to change each variable of a binomial  $b$  to a set of balanced variables, notice that the  $B$ -degree of the binomial is preserved. Hence it is enough to prove that if  $b = y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$  is a balanced binomial of  $B$ -degree  $n$  then  $b \in J_M$ .

With a balanced monomial  $m = y_{B_1} \cdots y_{B_n}$  we associate a bipartite multigraph  $G(m)$  with vertex classes  $B$  and  $E \setminus B$ , where  $E$  is the ground set of a matroid  $M$ . For each  $i$  we put an edge  $\{e, f\}$  in  $G(m)$  for  $f \in B, e \in E \setminus B$  if  $B_i = B \cup e \setminus f$ .

Observe that a balanced binomial  $b = y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n}$  belongs to  $I_M$  if and only if each vertex from  $E$  has the same degree with respect to  $G(y_{B_1} \cdots y_{B_n})$  and  $G(y_{D_1} \cdots y_{D_n})$ . Thus we can apply the following lemma, which is obvious from graph theory.

LEMMA 5. *Let  $G$  and  $H$  be two bipartite multigraphs with the same vertex classes. Suppose that each vertex has the same degree with respect to  $G$  and  $H$ . Then the symmetric difference of the multiset of edges of  $G$  and of  $H$  can be partitioned into even cycles, such that in each cycle two consecutive edges belong to different graph.*

We get a cycle  $f_1, e_1, f_2, e_2, \dots, f_r, e_r, f_1$ , such that for each  $i$  holds  $B'_i = B \cup e_i \setminus f_i \in \mathfrak{B}$ ,  $D'_i = B \cup e_{i-1} \setminus f_i \in \mathfrak{B}$  (with a cyclic numeration modulo  $r$ ),  $y_{B'_1} \cdots y_{B'_r}$  divides  $y_{B_1} \cdots y_{B_n}$  and  $y_{D'_1} \cdots y_{D'_r}$  divides  $y_{D_1} \cdots y_{D_n}$ .

Suppose  $r < n$ . Clearly, the balanced binomial  $b' = y_{B'_1} \cdots y_{B'_r} - y_{D'_1} \cdots y_{D'_r}$  belongs to  $I_M$  and has  $B$ -degree less than  $n$ . From inductive assumption  $b' \in J_M$ . There is equality

$$b = y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} = m_1 b' - y_{D'_1} \cdots y_{D'_r} (m_2 - m_3)$$

for some balanced monomials  $m_1, m_2, m_3$  such that  $m_2 - m_3 \in I_M$ . Balanced binomial  $b'' = m_2 - m_3 \in I_M$  has  $B$ -degree less than  $n$ , thus by inductive assumption  $b'' \in J_M$ , and as a consequence  $b \in J_M$ .

Suppose now that  $r = n$ . We can assume also that  $E = \{e_1, \dots, e_n, f_1, \dots, f_n\}$ , since otherwise we can contract  $B \setminus \{f_1, \dots, f_n\}$  and restrict our matroid to the set  $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ . Obviously the property of being generated extends from such a minor to a matroid.

For monomials  $m_1, m_2$  we say that  $m_2$  is *achievable* from  $m_1$  if  $m_1 - m_2 \in J_M$ , in this situation we say that variables of  $m_2$  are *achievable* from  $m_1$ . Observe that if there is a variable different from  $y_B$  that is achievable from both monomials  $y_{B_1} \cdots y_{B_n}$  and  $y_{D_1} \cdots y_{D_n}$ , then the assertion follows by induction. Indeed, if a variable  $y_{B'}$  is achievable from both, then there are monomials  $m_1, m_2$  such that

$$b = (y_{B_1} \cdots y_{B_n} - y_{B'} m_1) + (y_{B'} m_2 - y_{D_1} \cdots y_{D_n}) + y_{B'} (m_1 - m_2).$$

Binomial  $b' = m_1 - m_2 \in I_M$  has  $B$ -degree less than  $n$ , thus by inductive assumption  $y_B^{\deg_B(b') - \deg(b')} b' \in J_M$ . Then  $0 = \deg_B(b) - \deg(b) = \deg_B(y_{B'}) - \deg(y_{B'}) + \deg_B(b') - \deg(b')$ , so  $y_{B'} b' = (y_B^{\deg_B(y_{B'}) - \deg(y_{B'})} y_{B'}) \cdot (y_B^{\deg_B(b') - \deg(b')} b') \in J_M$ , which gives the assertion.

Suppose contrary – no variable different from  $y_B$  is achievable from both monomials of  $b$ , and we will reach a contradiction. For  $k, i \in \mathbb{Z}_n$  we define:

$$S_k^i := B \cup \{e_k, e_{k+1}, \dots, e_{k+i-1}\} \setminus \{f_k, f_{k+1}, \dots, f_{k+i-1}\},$$

$$T_k^i := B \cup \{e_{k-1}, e_k, \dots, e_{k+i-2}\} \setminus \{f_k, f_{k+1}, \dots, f_{k+i-1}\},$$

$$U_k^i := B \cup \{e_{k-i}\} \setminus \{f_k\}.$$

Sets  $S_k^i$  and  $T_k^i$  differ only on the set  $\{e_1, \dots, e_n\}$  by a shift by one. Notice that  $S_k^n = T_{k'}^n$  for arbitrary  $k, k' \in \mathbb{Z}_n$  and  $S_k^1 = U_k^0 = B'_i$ ,  $T_k^1 = U_k^1 = D'_i$ . Hence  $m_1 = y_{B'_1} \cdots y_{B'_n} = y_{S_1^1} \cdots y_{S_n^1}$  and  $m_2 = y_{D'_1} \cdots y_{D'_n} = y_{T_1^1} \cdots y_{T_n^1}$  are monomials of  $b$ .

LEMMA 6. *Suppose that for a fixed  $0 < i < n$  and all  $k \in \mathbb{Z}_n$  the following conditions hold:*

- (1) *sets  $S_k^i$  and  $T_k^i$  are bases,*
- (2) *monomial  $y_B^{i-1} y_{S_k^i} \prod_{j \neq k, \dots, k+i-1} y_{S_j^1}$  is achievable from  $m_1$ ,*
- (3) *monomial  $y_B^{i-1} y_{T_k^i} \prod_{j \neq k, \dots, k+i-1} y_{T_j^1}$  is achievable from  $m_2$ .*

*Then for all  $k \in \mathbb{Z}_n$  neither of the sets  $U_k^{-i}$ ,  $U_k^{i+1}$  is a basis.*

PROOF. Suppose contrary that  $U_k^{-i}$  is a basis. Binomial  $y_{S_k^1} y_{S_{k+1}^i} = y_{T_{k+1}^i} y_{U_k^{-i}}$  by the definition belongs to  $J_M$ . Thus, by the assumption, variable  $y_{T_{k+1}^i}$  would be achievable from both  $m_1$  and  $m_2$ , which is a contradiction. The argument for  $U_l^{i+1}$  is analogous, it differs by a shift by one.  $\square$

LEMMA 7. *Suppose that for a fixed  $0 < i < n$  and all  $k \in \mathbb{Z}_n$  the following conditions hold:*

- (1) *sets  $S_k^i$  are bases,*
- (2) *monomial  $y_B^{i-1} y_{S_k^i} \prod_{j \neq k, \dots, k+i-1} y_{S_j^1}$  is achievable from  $m_1$ ,*
- (3) *set  $U_k^{-j}$  is not a basis for any  $0 < j \leq i$ .*

Then for any  $k \in \mathbb{Z}_n$  the set  $S_k^{i+1}$  is a basis and the monomial  $y_B^i y_{S_k^{i+1}} \prod_{j \neq k, \dots, k+i} y_{S_j^1}$  is achievable from  $m_1$ .

PROOF. From the symmetric exchange property for  $e_k \in S_k^1$  follows that there exists  $x \in S_{k+1}^i \setminus S_k^1$  such that  $\tilde{S}_{k+1}^i = S_{k+1}^i \cup e_k \setminus x$  and  $\tilde{S}_k^1 = S_k^1 \cup x \setminus e_k$  are also bases. Thus  $x \in \{f_k, e_{k+1}, e_{k+2}, \dots, e_{k+i}\}$ . Notice that if  $x = e_{k+j}$  for some  $j$ , then  $\tilde{S}_k^1 = U_k^{-j}$ . Thus, by condition (3), we get that  $x = f_k$ . This means that  $\tilde{S}_{k+1}^i = S_k^{i+1}$  and  $\tilde{S}_k^1 = B$ . In particular the binomial  $y_{S_{k+1}^i} y_{S_k^1} - y_B y_{S_k^{i+1}}$  belongs to  $J_M$  (condition (1) guarantees that variable  $y_{S_{k+1}^i}$  exists). Thus, by condition (2) follows the assertion.  $\square$

Analogously we get the following shifted version of the above lemma.

LEMMA 8. Suppose that for a fixed  $0 < i < n$  and all  $k \in \mathbb{Z}_n$  the following conditions hold:

- (1) sets  $T_k^i$  are bases,
- (2) monomial  $y_B^{i-1} y_{T_k^i} \prod_{j \neq k, \dots, k+i-1} y_{T_j^1}$  is achievable from  $m_2$ ,
- (3) set  $U_k^{j+1}$  is not a base for any  $0 < j \leq i$ .

Then for any  $k \in \mathbb{Z}_n$  the set  $T_k^{i+1}$  is a basis and the monomial  $y_B^i y_{T_k^{i+1}} \prod_{j \neq k, \dots, k+i} y_{T_j^1}$  is achievable from  $m_2$ .  $\square$

We are ready to reach a contradiction by an inductive argument. First we verify that for  $i = 1$  the assumptions of Lemma 6 are satisfied. Suppose now that for some  $1 < i < n$  the assumptions of Lemma 6 are satisfied for all  $1 \leq j \leq i$ . Then, by Lemma 6 the assumptions of both Lemma 7 and Lemma 8 are satisfied for all  $1 \leq j \leq i$ . Thus by the assertion of Lemmas 7 and 8, the assumptions of Lemma 6 are satisfied for all  $1 \leq j \leq i + 1$ . We obtain that the assumptions of Lemmas 6, 7 and 8 are satisfied for all  $1 \leq i < n$ . For  $i = n - 1$  we obtain that the monomial  $y_B^{n-1} y_{S_1^n} = y_B^{n-1} y_{T_1^n}$  is achievable from both  $m_1$  and  $m_2$ , this gives a contradiction.  $\square$

#### 4. Remarks

We begin with the original formulation of conjectures stated by White in [23].

Two sequences of bases  $\mathcal{B} = (B_1, \dots, B_n)$  and  $\mathcal{D} = (D_1, \dots, D_n)$  are *compatible* if  $B_1 \cup \dots \cup B_n = D_1 \cup \dots \cup D_n$  as multisets (that is if  $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ ).

White defines three equivalence relations. Two sequences of bases  $\mathcal{B}$  and  $\mathcal{D}$  of equal length are in relation:

- $\sim_1$  if  $\mathcal{D}$  may be obtained from  $\mathcal{B}$  by a composition of symmetric exchanges. That is  $\sim_1$  is the transitive closure of the relation which exchanges a pair of bases  $B_i, B_j$  in a sequence into a pair obtained by a double swap.
- $\sim_2$  if  $\mathcal{D}$  may be obtained from  $\mathcal{B}$  by a composition of symmetric exchanges and permutations of the order of the bases.
- $\sim_3$  if  $\mathcal{D}$  may be obtained from  $\mathcal{B}$  by a composition of multiple symmetric exchanges.

Let  $TE(i)$  denote the class of matroids such that for all  $n \geq 2$  and for all compatible sequences of bases  $\mathcal{B}, \mathcal{D}$  of length  $n$  holds  $\mathcal{B} \sim_i \mathcal{D}$ . An algebraic meaning of the property  $TE(3)$  is that the toric ideal  $I_M$  is generated by quadratic binomials.

The property  $TE(2)$  means that the toric ideal  $I_M$  is generated by quadratic binomials corresponding to double swaps, while the property  $TE(1)$  is its analog for noncommutative polynomial ring  $S_M$ .

We are ready to formulate the original conjecture [23, Conjecture 12] of White.

CONJECTURE 9. *The following equalities hold:*

- (1)  $TE(1) = \text{the class of all matroids}$ ,
- (2)  $TE(2) = \text{the class of all matroids}$ ,
- (3)  $TE(3) = \text{the class of all matroids}$ .

Clearly, Conjecture 1 coincides with Conjecture 9 (2). It is straightforward [23, Proposition 5] that:

- (1)  $TE(1) \subset TE(2) \subset TE(3)$ ,
- (2) classes  $TE(1)$ ,  $TE(2)$  and  $TE(3)$  are closed under taking minors and dual matroid,
- (3) classes  $TE(1)$  and  $TE(3)$  are closed under direct sum.

White claims also that the class  $TE(2)$  is closed under direct sum, however unfortunately there is a gap in his proof. We will show some relations between classes  $TE(1)$  and  $TE(2)$ .

LEMMA 10. *For a matroid  $M$  the following conditions are equivalent:*

- (1)  $M \in TE(1)$ ,
- (2)  $M \in TE(2)$  and for any two bases  $B_1, B_2$  of  $M$  holds  $(B_1, B_2) \sim_1 (B_2, B_1)$ .

PROOF. Implication (1)  $\Rightarrow$  (2) is clear from the definition. To get another implication it is enough to notice that any permutation is a composition of transpositions.  $\square$

Above lemma allows us to make a link with a so-called cyclic ordering conjecture. It was first suspected by Gabow [9], and later formulated as a conjecture by Kajitani, Ueno and Miyano [12].

CONJECTURE 11. *Suppose  $M$  is a matroid whose ground set is a union of two disjoint bases. Then it is possible to place elements of the ground set of  $M$  on a circle, such that any  $r(M)$  consecutive elements form a basis of  $M$ .*

Cordovil and Moreira [6] proved it for graphical matroids (see also [25] for another proof, and [11] for other partial results). Observe that Conjecture 11 implies that for some two bases  $B_1, B_2$  of a matroid holds  $(B_1, B_2) \sim_1 (B_2, B_1)$ . This together with Lemma 10 gives that  $TE(1) = TE(2)$ .

PROPOSITION 12. *For a matroid  $M$  the following conditions are equivalent:*

- (1)  $M \in TE(1)$ ,
- (2)  $M \oplus M \in TE(1)$ ,
- (3)  $M \oplus M \in TE(2)$ .

PROOF. Implications (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) were already discussed. To get (3)  $\Rightarrow$  (1) suppose that a matroid  $M$  satisfies  $M \oplus M \in TE(2)$ .

First we prove that  $M \in TE(2)$ . Let  $\mathcal{B} = (B_1, \dots, B_n)$  and  $\mathcal{D} = (D_1, \dots, D_n)$  be compatible sequences of bases of  $M$ . If  $B$  is a basis of  $M$  then  $\mathcal{B}' = ([B_1, B], \dots)$  and  $\mathcal{D}' = ([D_1, B], \dots)$  are compatible sequences of bases of  $M \oplus M$ , where  $[B', B'']$  denotes a basis of  $M \oplus M$  consisting of a basis  $B'$  of  $M$  on the first copy and  $B''$  on



the second. By the assumption  $\mathcal{B}' \sim_2 \mathcal{D}'$ . Notice that any symmetric exchange in  $M \oplus M$  on the bases appearing as first coordinates is either trivial or is a symmetric exchange. Thus, the same symmetric exchanges certify that  $\mathcal{B} \sim_2 \mathcal{D}$  in  $M$ .

Due to Lemma 10 in order to prove  $M \in TE(1)$  it is enough to show that for any two bases  $B_1, B_2$  of  $M$  holds  $(B_1, B_2) \sim_1 (B_2, B_1)$ . Sequences of bases  $([B_1, B_1], [B_2, B_2])$  and  $([B_2, B_1], [B_1, B_2])$  in  $M \oplus M$  are compatible. Thus by the assumption one can be obtained from the other by a composition of symmetric exchanges and permutations. By the symmetry, without loss of generality we can assume that one did not use a permutation. Now the projection of this symmetric exchanges to the first copy shows that  $(B_1, B_2) \sim_1 (B_2, B_1)$  in  $M$ .  $\square$

As a corollary we get that for reasonable classes of matroids the 'standard' version of White's conjecture is equivalent to the 'strong' one.

**COROLLARY 13.** *If a class of matroids  $\mathfrak{C}$  is closed under direct sums, then  $\mathfrak{C} \subset TE(1)$  if and only if  $\mathfrak{C} \subset TE(2)$ . In particular, strongly base orderable, graphical, and cographical matroids belong to  $TE(1)$ . Additionally, Conjectures 9 (1) and (2) are equivalent.*

Moreover, we get that  $TE(2)$  is closed under direct sum if and only if  $TE(1) = TE(2)$ , which we believe is an open question.

White states also an intermediate conjecture [23, Conjecture 13], saying that relations  $\sim_1$  and  $\sim_3$  are equal. That is every multiple symmetric exchange is a composition of symmetric exchanges. In other words, for every bases  $B_1, B_2$  and  $A_1 \subset B_1, A_2 \subset B_2$ , such that both  $B'_1 = B_1 \cup A_2 \setminus A_1$  and  $B'_2 = B_2 \cup A_1 \setminus A_2$  are bases one can obtain  $(B'_1, B'_2)$  from  $(B_1, B_2)$  by a composition of symmetric exchanges.

It would be very interesting to prove even a weaker statement saying that for every bases  $B_1, B_2$  and  $A_1 \subset B_1$  there exists  $A_2 \subset B_2$ , such that both  $B'_1 = B_1 \cup A_2 \setminus A_1$  and  $B'_2 = B_2 \cup A_1 \setminus A_2$  are bases and one can obtain  $(B'_1, B'_2)$  from  $(B_1, B_2)$  by a composition of symmetric exchanges. This would imply that  $TE(1) = TE(2)$ . White claims that Brylawski showed it in [3], however unfortunately such a statement does not follow from Brylawski's argument.

In the same way as we associate the toric ideal with a matroid one can associate a toric ideal  $I_P$  with a discrete polymatroid  $P$ . Herzog and Hibi [10] extend White's conjecture to discrete polymatroids, they also ask if the toric ideal  $I_P$  of a discrete polymatroid possesses a quadratic Gröbner basis (we refer the reader to [21]).

**REMARK 14.** *Theorem 2 and Theorem 3 are true for discrete polymatroids.*

There are several ways to prove that our results hold also for discrete polymatroids. One possibility is to use Lemma 5.4 from [10] to reduce a question if a binomial is generated by quadratic binomials corresponding to single element symmetric basis exchanges from a discrete polymatroid to a certain matroid. Another possibility is to associate to a discrete polymatroid  $P \subset \mathbb{Z}^n$  a matroid  $M_P$  on the ground set  $\{1, \dots, r(P)\} \times \{1, \dots, n\}$  in which a set  $I$  is independent if there is  $v \in P$  such that for all  $i$  holds  $|I \cap \{1, \dots, r(P)\} \times \{i\}| \leq v_i$ . It is straightforward that compatibility of sequences of bases and generation are the same in  $P$  and in  $M_P$ . Moreover, one can easily prove that a double swap in  $M_P$  corresponds to at most two double swaps in  $P$ .

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